

CHARACTERIZATION OF DISTRIBUTIONS HAVING A VALUE AT A POINT IN THE SENSE OF ROBINSON

HANS VERNAEVE AND JASSON VINDAS

ABSTRACT. We characterize Schwartz distributions having a value at a single point in the sense introduced by means of nonstandard analysis by A. Robinson. They appear to be distributions continuous in a neighborhood of the point.

1. INTRODUCTION

In [9, §5.3], A. Robinson initiated the use of nonstandard analysis in the theory of Schwartz distributions. Among other things, he introduces nonstandard representatives of a Schwartz distribution and, by means of a property of the representatives, he introduces a notion of point value of a distribution.

A natural question to raise is how Robinson's notion of point value is related to the classical definition of point value in the sense of Łojasiewicz [5, 6]. Through investigation of this question, we arrived at a characterization which is the main result of this paper: a distribution has a value at $x_0 \in \Omega$ in the sense of Robinson iff it is a continuous function in a neighborhood of x_0 . Our characterization improves an earlier result of P. Loeb [4] which has to assume the everywhere existence of Robinson point values (cf. section 3).

We mention that the concept of point value in the sense of Łojasiewicz has shown to be of great importance in several areas of mathematical analysis and its applications, such as spectral expansions [1, 11, 14], sampling theorems and summability of cardinal series [12], edge detection from spectral data [2], or the convergence of wavelet expansions [8, 13].

2. NOTATIONS

By Ω , we always denote an open subset of \mathbb{R}^n . We denote $B(a, r) := \{x \in \mathbb{R}^n : |x - a| < r\}$.

2.1. Schwartz distributions. We denote by $\mathcal{D}(\Omega)$ the space of functions in $\mathcal{C}^\infty(\Omega)$ with compact support contained in Ω . Its dual space $\mathcal{D}'(\Omega)$ is the space of Schwartz distributions on Ω . We denote the action of a linear map $T: \mathcal{D}(\Omega) \rightarrow \mathbb{C}$ on an element $\phi \in \mathcal{D}(\Omega)$ by means of the pairing $\langle T, \phi \rangle$. Sometimes it is useful to denote functions and distributions by means of its action on a dummy variable (e.g., as in [1]); we then denote the action of T on ϕ as $\langle T(x), \phi(x) \rangle$. This allows us to write changes of variables $y = F(x)$ simply as $\langle T(F(x)), \phi(x) \rangle = \langle T(y), \phi(F^{-1}(y)) \cdot |DF^{-1}(y)| \rangle$. We refer to [1, 10] for further information about Schwartz distributions.

2000 *Mathematics Subject Classification.* 26E35, 46F10, 46S20.

Key words and phrases. Schwartz distributions, nonstandard analysis, point values.

J. Vindas gratefully acknowledges support by a Postdoctoral Fellowship of the Research Foundation–Flanders (FWO, Belgium).

2.2. Nonstandard analysis. We work in the framework of nonstandard analysis as introduced by Robinson [9]. We refer to [3] for a more accessible introduction to nonstandard analysis. As usual, if $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a function, we keep the notation f for its canonical extension *f (and similarly for relations on \mathbb{R}^n). Also ${}^*\int$ will be denoted by \int . We denote the set of all finite numbers in ${}^*\mathbb{C}$ by $\text{Fin}({}^*\mathbb{C})$ and write $x \approx y$ if $|x - y|$ is infinitesimal ($x, y \in {}^*\mathbb{R}^n$). We write $x \lesssim y$ if $x \leq y$ or $x \approx y$ ($x, y \in {}^*\mathbb{R}$). We denote the standard part (a.k.a. shadow) by st .

3. KNOWN RESULTS

Robinson works with real valued distributions on the real line, but the generalization to complex valued distributions on an open subset Ω of \mathbb{R}^n is in most cases straightforward. We say that a function $f \in {}^*\mathcal{C}^\infty(\Omega)$ represents (in the sense of Robinson) a (not necessarily continuous) linear map $T: \mathcal{D}(\Omega) \rightarrow \mathbb{C}$ if $\int_\Omega f \phi \approx \langle T, \phi \rangle$ for each $\phi \in \mathcal{D}(\Omega)$. In fact, more general functions than ${}^*\mathcal{C}^\infty(\Omega)$ -functions are allowed as representatives, but ${}^*\mathcal{C}^\infty(\Omega)$ -functions suffice to develop distribution theory by infinitesimal means. Robinson calls equivalence classes of functions representing the same map T predistributions. We will identify the predistribution with the map T .

Robinson calls a predistribution standard at $x_0 \in \Omega$ if it has a representative f that is S-continuous at x_0 , i.e., such that $f(x) \approx f(x_0)$ for each $x \approx x_0$ [9, p. 140]. He shows (with a slightly different proof):

Theorem 3.1 (Robinson). *Let T be a linear map $\mathcal{D}(\Omega) \rightarrow \mathbb{C}$. If T has a representative f that is S-continuous at $x_0 \in \Omega$, then $f(x_0) \in \text{Fin}({}^*\mathbb{C})$. Moreover, the value $\text{st } f(x_0)$ does not depend on the chosen S-continuous representative.*

Proof. Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Since f is internal and S-continuous, we find by overspill (see e.g. [3, 11.9.1]) on the set

$$\{r \in {}^*\mathbb{R}, r > 0 : (\forall x \in {}^*\Omega) |x - x_0| \leq r \implies |f(x) - f(x_0)| \leq \varepsilon\}$$

that there exists $r \in \mathbb{R}$, $r > 0$ such that $|f(x) - f(x_0)| \leq \varepsilon$ for each $x \in {}^*B(x_0, r) \subseteq {}^*\Omega$. Now let $\phi \in \mathcal{D}(B(x_0, r))$ with $\int_\Omega \phi = 1$. Let $C := \int_\Omega |\phi| \in \mathbb{R}$. Then, since f represents T ,

$$\begin{aligned} |\langle T, \phi \rangle - f(x_0)| &\approx \left| \int_\Omega f(x) \phi(x) dx - f(x_0) \right| = \left| \int_{{}^*B(x_0, r)} (f(x) - f(x_0)) \phi(x) dx \right| \\ &\leq \int_{{}^*B(x_0, r)} \varepsilon |\phi(x)| dx = C\varepsilon. \end{aligned}$$

In particular, $f(x_0) \in \text{Fin}({}^*\mathbb{C})$. For any g representing T and S-continuous at x_0 , we have the same inequality (possibly only for some smaller $r \in \mathbb{R}$, $r > 0$), so $|f(x_0) - g(x_0)| \lesssim 2C\varepsilon$. As ε is arbitrary, $\text{st } f(x_0) = \text{st } g(x_0)$. \square

The number $\text{st } f(x_0)$ is called the value (in the sense of Robinson) of the predistribution at x_0 . P. Loeb [4] proves that if T admits a value $g(x)$ at each $x \in \Omega$, then the resulting map $g: \Omega \rightarrow \mathbb{C}$ is continuous. In that case, *g represents T [9, 5.3.15], and hence T is a continuous function (as a regular Schwartz distribution). Actually, Loeb's result is a particular case of theorem 4.5, shown below.

A distribution $T \in \mathcal{D}'(\Omega)$ admits the value $c \in \mathbb{C}$ at $x_0 \in \Omega$ in the sense of Łojasiewicz [5, 6] if $\lim_{\varepsilon \rightarrow 0} T(x_0 + \varepsilon x) = c$, where the limit is interpreted in the distributional sense, i.e., if

$$\lim_{\varepsilon \rightarrow 0} \langle T(x_0 + \varepsilon x), \phi(x) \rangle = \langle c, \phi \rangle = c \int_\Omega \phi, \quad \forall \phi \in \mathcal{D}(\Omega).$$

Observe that if T is continuous in a neighborhood of x_0 , then one readily verifies that T has a Lojasiewicz value at x_0 and its value is in fact $T(x_0) = c$. On the other hand, the converse is not true, in general, as shown by the function $T(x) = |x|^{-\frac{1}{2}} e^{i/x}$, which is unbounded at the origin but it admits the Lojasiewicz value 0 at $x_0 = 0$. More generally, any function of the form $|x|^{-\beta} e^{i/|x|^\alpha}$, where $\alpha, \beta > 0$, uniquely determines a distribution that has Lojasiewicz value 0 at the origin [5].

Of crucial importance for our result is the following theorem [6, §6.2]:

Theorem 3.2. *Let $T \in \mathcal{D}'(\Omega)$ and $x_0 \in \Omega$. If $\lim_{\varepsilon \rightarrow 0, \lambda \rightarrow 0} T(x_0 + \varepsilon x + \lambda)$ exists in the distributional sense, i.e., $\lim_{\varepsilon \rightarrow 0, \lambda \rightarrow 0} \langle T(x_0 + \varepsilon x + \lambda), \phi(x) \rangle$ exists $\forall \phi \in \mathcal{D}(\Omega)$, then T is a continuous function in a neighborhood of x_0 .*

4. NEW RESULTS

Lemma 4.1. *Let T be a predistribution that is standard at $x_0 \in \Omega$. Then T is a Schwartz distribution in an open neighborhood ω of x_0 (i.e., $T|_\omega$ is a continuous map $\mathcal{D}(\omega) \rightarrow \mathbb{C}$).*

Proof. Let f be a representative of T that is S-continuous. As in the proof of theorem 3.1, there exists $r \in \mathbb{R}$, $r > 0$, such that $|f(x) - f(x_0)| \leq 1$ for each $x \in {}^*B(x_0, r) \subseteq {}^*\Omega$. Let $\phi \in \mathcal{D}(B(x_0, r))$. Then

$$\begin{aligned} |\langle T, \phi \rangle| &\approx \left| \int_{{}^*B(x_0, r)} f \phi \right| \leq \sup_{B(x_0, r)} |f| \mu(B(x_0, r)) \sup_{B(x_0, r)} |\phi| \\ &\leq (|f(x_0)| + 1) \mu(B(x_0, r)) \sup_{B(x_0, r)} |\phi|, \end{aligned}$$

where μ denotes the Lebesgue measure. Taking standard parts,

$$|\langle T, \phi \rangle| \leq (|\text{st } f(x_0)| + 1) \mu(B(x_0, r)) \sup_{B(x_0, r)} |\phi|.$$

Hence $T|_{B(x_0, r)}$ is continuous. □

Definition. We call $\psi \in {}^*\mathcal{D}(\mathbb{R}^n)$ a strict nonstandard delta function (this name corresponds to the standard notion of a *strict delta net*, see e.g. [7, §7]) if

- (1) $\int_{{}^*\mathbb{R}^n} \psi = 1$
- (2) $\psi(x) = 0, \forall x \not\approx 0$
- (3) $\int_{{}^*\mathbb{R}^n} |\psi| \in \text{Fin}({}^*\mathbb{R})$.

Lemma 4.2. *Let ψ be a strict nonstandard delta function. Let Ω be a (standard) neighborhood of 0 and let $f \in {}^*\mathcal{C}^0(\Omega)$ be S-continuous at 0. Then $\int_{{}^*\mathbb{R}^n} \psi f \approx f(0)$.*

Proof. As ψ is internal, there exists $r \in {}^*\mathbb{R}$, $r \approx 0$ such that $\psi(x) = 0$ if $|x| \geq r$ by overspill. Then

$$\left| f(0) - \int_{{}^*\mathbb{R}^n} \psi f \right| = \left| \int_{{}^*\mathbb{R}^n} (f(0) - f(x)) \psi(x) dx \right| \leq \sup_{|x| \leq r} |f(0) - f(x)| \int_{{}^*\mathbb{R}^n} |\psi| \approx 0.$$

□

In particular, each strict nonstandard delta function is a representative of the delta distribution, since ${}^*\phi$ is S-continuous for each $\phi \in \mathcal{D}(\mathbb{R}^n)$ (see e.g. [3, §7.1]).

Theorem 4.3. *Let T be a predistribution that admits the value c (in the sense of Robinson) at $x_0 \in \Omega$. Then $\langle {}^*T(x), \psi(x - x_0) \rangle \approx c$ for each strict delta function ψ .*

Proof. Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. In the proof of theorem 3.1, we showed that there exists $r \in \mathbb{R}$, $r > 0$ such that for each $\phi \in \mathcal{D}(B(x_0, r))$ with $\int_{\Omega} \phi = 1$ and $\int_{\Omega} |\phi| = C$, we have $|\langle T, \phi \rangle - c| \leq (C+1)\varepsilon$. By overspill, for each $\phi \in {}^*\mathcal{D}(B(x_0, r))$ with $\int_{*\Omega} \phi = 1$ and $\int_{*\Omega} |\phi| = C$, we have $|\langle {}^*T, \phi \rangle - c| \leq (C+1)\varepsilon$. This holds in particular for any $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ and $\phi(x) = \psi(x - x_0)$ with ψ a strict nonstandard delta function. Hence $\langle {}^*T(x), \psi(x - x_0) \rangle \approx c$ for such ψ . \square

From our main result, theorem 4.5, it will follow that the property in the statement of the previous theorem actually characterizes predistributions admitting a value at x_0 in the sense of Robinson.

The previous theorem sheds a light on the relation between point values in the sense of Robinson and point values in the sense of Łojasiewicz. By a nonstandard characterization of limits (see e.g. [3, §7.3]) a distribution $T \in \mathcal{D}'(\Omega)$ admits the value c at $x_0 \in \Omega$ in the sense of Łojasiewicz iff $\langle {}^*T(x), \psi(x - x_0) \rangle \approx c$ for each so-called model delta function ψ , i.e., $\psi(x) = \rho^{-n}\phi(x/\rho)$ with $\phi \in \mathcal{D}(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \phi = 1$, $\rho \in {}^*\mathbb{R} \setminus \{0\}$, $\rho \approx 0$ (the name model delta function corresponds to the standard notion of a *model delta net*, see e.g. [7, §7]).

Lemma 4.4. *Let ω be an open subset of Ω and let $x_0 \in \omega$. Then a linear map $T: \mathcal{D}(\Omega) \rightarrow \mathbb{C}$ admits the value c at x_0 (in the sense of Robinson) iff $T|_{\omega}: \mathcal{D}(\omega) \rightarrow \mathbb{C}$ admits the value c at x_0 (in the sense of Robinson).*

Proof. \Rightarrow : immediate.

\Leftarrow : Let $f \in {}^*\mathcal{C}^\infty(\omega)$ be a representative of $T|_{\omega}$, i.e., $\int_{*\omega} f\phi \approx \langle T, \phi \rangle$, $\forall \phi \in \mathcal{D}(\omega)$, and suppose that f is S-continuous in x_0 . Let $g \in {}^*\mathcal{C}^\infty(\Omega)$ be any representative of T . Then $f - g|_{\omega}$ is a representative of the 0-distribution on ω . Let $\chi \in \mathcal{D}(\omega)$ with $\chi = 1$ on some (standard) neighborhood V of x_0 . Then $\int_{*\Omega} (f - g)\chi\phi \approx 0$, $\forall \phi \in \mathcal{D}(\Omega)$, since $\chi\phi \in \mathcal{D}(\omega)$. So $(f - g)\chi$ is a representative of 0 on Ω , and $f\chi + g(1 - \chi)$ is a representative of T on Ω which is equal to f in a (standard) neighborhood of x_0 . \square

Theorem 4.5. *Let T be a predistribution and $x_0 \in \Omega$. Then T is standard at x_0 iff T is a continuous function in a neighborhood of x_0 .*

Proof. \Rightarrow : by lemma 4.1, there exists an open neighborhood ω of x_0 such that $T \in \mathcal{D}'(\omega)$. Let c denote the value of T at x_0 (in the sense of Robinson). Let $\phi \in \mathcal{D}(\omega)$ with $\int_{\omega} \phi = 1$. Let $\varepsilon \in {}^*\mathbb{R}$, $\varepsilon \approx 0$ and $\lambda \in {}^*\mathbb{R}^n$, $\lambda \approx 0$. Then, by theorem 4.3,

$$\langle {}^*T(x_0 + \varepsilon x + \lambda), \phi(x) \rangle = \left\langle {}^*T(x), \varepsilon^{-n} \phi\left(\frac{x - x_0 - \lambda}{\varepsilon}\right) \right\rangle \approx c,$$

since $\psi(x) := \varepsilon^{-n} \phi\left(\frac{x - \lambda}{\varepsilon}\right) \in {}^*\mathcal{D}(\mathbb{R}^n)$ is a strict nonstandard delta function:

1. by a change of variables, $\int_{*\mathbb{R}^n} \psi = \int_{*\mathbb{R}^n} \phi = 1$.
2. $\psi(x) \neq 0$ iff $\frac{x - \lambda}{\varepsilon} \in \text{supp } \phi$ iff $x \in \lambda + \varepsilon \text{supp } \phi$; then in particular $x \approx 0$.
3. $\int_{*\mathbb{R}^n} |\psi| = \int_{*\mathbb{R}^n} |\phi| \in \mathbb{R}$ is finite.

As ε and λ are arbitrary, by a nonstandard characterization of limits (see e.g. [3, §7.3])

$$\lim_{\lambda \rightarrow 0, \varepsilon \rightarrow 0} \langle T(x_0 + \varepsilon x + \lambda), \phi(x) \rangle = c$$

for each $\phi \in \mathcal{D}(\omega)$ with $\int_{\omega} \phi = 1$. By theorem 3.2, T is a continuous function in a neighborhood of x_0 .

\Leftarrow : let ω be an open neighborhood of x_0 and $f \in \mathcal{C}^0(\omega)$ such that $T = f$ on ω . Let $g := f \star \psi \in {}^*\mathcal{C}^\infty(\omega')$ for some neighborhood ω' of x_0 (\star denoting convolution), where ψ is a strict nonstandard delta function. By lemma 4.2, $g(x) = \int_{*\mathbb{R}^n} f(x - y)\psi(y) dy \approx f(x)$ for each $x \in {}^*\omega'$ (lemma 4.2 can be applied since f is

S-continuous at any $x \in {}^*\omega'$ if the closure of ω' is contained in ω , see e.g. [3, §7.1]). In particular, g is S-continuous at x_0 . Then for any $\phi \in \mathcal{D}(\omega')$,

$$\left| \int_{{}^*\omega'} g\phi - \langle T, \phi \rangle \right| = \left| \int_{{}^*\omega'} g\phi - \int_{{}^*\omega'} f\phi \right| \leq \sup_{{}^*\omega'} |g - f| \int_{{}^*\omega'} |\phi| \approx 0,$$

so g represents T on ω' . By lemma 4.4, T admits the value $\text{st } g(x_0)$ at x_0 . \square

REFERENCES

- [1] R. Estrada, R. Kanwal, A Distributional Approach to Asymptotics, 2nd edition. Birkhäuser, Boston, 2002.
- [2] R. Estrada, J. Vindas, Determination of jumps of distributions by differentiated means, Acta Math. Hungar. 124 (2009) 215–241.
- [3] R. Goldblatt, Lectures on the Hyperreals. An introduction to Nonstandard Analysis. Graduate Texts in Math. vol. 188. Springer, New York, 1998.
- [4] P. Loeb, A note on continuity for Robinson's predistributions. In *Victoria Symposium on Nonstandard Analysis*, Lecture Notes in Math. 369, 153–154. Springer Verlag, Berlin, 1974.
- [5] S. Łojasiewicz, Sur la valeur et la limite d'une distribution en un point, Studia Math. 16 (1957) 1–36.
- [6] S. Łojasiewicz, Sur la fixation des variables dans une distribution, Studia Math. 17 (1958) 1–64.
- [7] M. Oberguggenberger, Multiplication of Distributions and Applications to Partial Differential Equations. Pitman Res. Not. Math. 259. Longman Sci. Techn., Essex, 1992.
- [8] S. Pilipović, N. Teofanov, Multiresolution expansion, approximation order and quasiasymptotic behavior of tempered distributions, J. Math. Anal. Appl. 331 (2007) 455–471.
- [9] A. Robinson, Non-Standard Analysis. North-Holland, Amsterdam, 1966.
- [10] L. Schwartz, Théorie des distributions. Hermann, Paris, 1966.
- [11] G. G. Walter, Pointwise convergence of distribution expansions, Studia Math. 26 (1966) 143–154.
- [12] G. G. Walter, Sampling bandlimited functions of polynomial growth, SIAM J. Math. Anal. 19 (1988) 1198–1203.
- [13] G. G. Walter, Pointwise convergence of wavelet expansions, J. Approx. Theory 80 (1995) 108–118.
- [14] J. Vindas, R. Estrada, Distributional point values and convergence of Fourier series and integrals, J. Fourier Anal. Appl. 13 (2007) 551–576.

Ghent University, Krijgslaan 281, B-9000 Gent, Belgium.
E-mail address: `hvernaev@cage.ugent.be`

Ghent University, Krijgslaan 281, B-9000 Gent, Belgium.
E-mail address: `jvindas@cage.ugent.be`